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# The $q$-deformed Krichever–Novikov algebra and Ward $q$-identities for correlators on higher genus Riemann surfaces

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**Abstract.** The $q$-deformed Krichever–Novikov algebra on higher genus Riemann surfaces is obtained by means of the operator product expansion method. The Ward $q$-identities for correlation functions of primary fields are derived. It is found that the Ward $q$-identities cannot determine the two-point correlation function.

## 1. Introduction

Over many years much attention has been paid to the Krichever–Novikov (KN) algebra [1], the underlying symmetry of the conformal field theories on higher genus Riemann surfaces [2–4]. The KN algebra is a generalization of the Virasoro algebra from a trivial Riemann surface to a higher-genus Riemann surface, which enables us to treat the Teichmuller deformation and conformal deformations on the same footing.

Consider a compact Riemann surface $\Sigma$ of genus $g$ with two distinguished points $P_+$ and $P_-$ in a general position; the KN algebra is defined by the relation

$$[L_n, L_m] = \sum_{r=-g_0}^{g_0} C^r_{nm} L_{n+m-r} \tag{1}$$

where $g_0 = \frac{3}{2}g$, and the structure constants are given by

$$C^r_{nm} = \oint_{C_\tau} \mathrm{d}w\, (e_n(w)e'_m(w) - e'_n(w)e_m(w))\Omega_{n+m-r}(w). \tag{2}$$

Here we have used the KN bases on $\Sigma$:

$$\begin{aligned} e_n(Q) &= z_\pm^{\pm n - g_0 + 1}(1 + O(z_\pm)) \frac{\partial}{\partial z_\pm} \\ \Omega_n(Q) &= z_\pm^{\mp n + g_0 - 2}(1 + O(z_\pm))(\mathrm{d}z_\pm)^2 \end{aligned} \tag{3}$$

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where $Q \in \Sigma$, and $z_\pm$ is the local coordinates in the neighbourhood of $P_\pm$. They satisfy the following duality relation:

$$\oint_{C_\tau} e_n(Q)\Omega_m(Q) = \delta_{nm} \tag{4}$$

where the contours $C_\tau = \{Q \in \Sigma, \tau(Q) = \tau\}$ are level lines of the univalent function

$$\tau(Q) = \mathrm{Re} \int_{Q_0}^{Q} \mathrm{d}p \tag{5}$$

where $\mathrm{d}p$ is the third kind of differential on $\Sigma$ with poles of first order at the points $P_\pm$ with residues $\pm 1$, $Q_0$ is an arbitrary initial point, and as $\tau \to \pm\infty$, $C_\tau$ become circles enveloping the points $P_\pm$.

During the last few years a growing interest in the study of quantum groups and algebras [5, 6] has appeared. These new mathematical objects play an important role in some quantum systems, such as exactly solved statistical models [7], integrable field theory [7], vertex and spin models [8] and conformal field theory [9]. Their applications in molecular, nuclear, particle physics and quantum optics [10–12] have also been investigated in recent years. More recently, a great deal of attention has been paid to the $q$-Virasoro algebra [13–21]. It is well known that the operator product expansion (OPE) method is an effective approach to study (super)conformal algebras. It is used widely to construct superconformal algebras on higher-genus Riemann surfaces [2–4]. Recently, it has also been applied to the $q$-Virasoro algebra [13–21]. In this paper, we intend to use the OPE method to obtain a $q$-deformation of the KN algebra, the $q$-KN algebra, and to derive the Ward $q$-identities for correlators on higher-genus Riemann surfaces.

## 2. A $q$-deformation of the KN algebra

The $q$-KN algebra on $\Sigma$ may be generated by the energy-momentum tensor $T(z)$ which can be expanded in terms of the KN bases on $\Sigma$ as

$$T(Q) = \sum_n L_n \Omega^n(Q) \tag{6}$$

where $L_n$ are the generators of the $q$-KN algebra. Using (5), we have

$$L_n = \oint_{C_\tau} T(Q)e_n(Q). \tag{7}$$

In a local complex coordinate $z$ that vanishes at the point $P_+$, equation (7) can be re-expressed as

$$L_n = \oint_{C_\tau} \mathrm{d}z\, T(z)e_n(z). \tag{8}$$

We now would like to evaluate the following bracket

$$[L_n, L_m] = (L_nL_m)_q - (L_mL_n)_q \tag{9}$$

where the terms $(\ )_q$ are defined via the $q$-product of two field operators $A(z)$ and $B(w)$ [19]:

$$(A(z)B(w))_q = A(zq)B(wq^{-1}). \tag{10}$$

For instance, we have

$$\begin{aligned}(L_nL_m)_q &= \oint_{C_\tau} \mathrm{d}z \oint_{C_0} \mathrm{d}w\, e_n(z)e_m(w)(T(z)T(w))_q \\ &= q^{m-n} \oint_{C_\tau} \mathrm{d}z \oint_{C_0} \mathrm{d}w\, e_n(qz)e_m(wq^{-1})T(zq)T(wq^{-1}) \\ &= q^{m-n}L_nL_m. \end{aligned} \tag{11}$$

Similarly we have

$$(L_mL_n) = q^{-(m-n)}L_mL_n. \tag{12}$$

Combining (11) and (12), we then can rewrite the bracket in (9) as

$$[L_n, L_m] = q^{m-n}L_nL_m - q^{-(m-n)}L_mL_n. \tag{13}$$

With the help of (8), (11) and (12), the above bracket can be expressed as a complex contour integral

$$\begin{aligned}[L_n, L_m] &= \oint_{C_\tau} \mathrm{d}z \oint_{C_w} \mathrm{d}w\, e_n(z)e_m(w)\{(T(z)T(w)_q - (T(w)T(z))_q\} \\ &= \oint_{C_\tau} \mathrm{d}z \oint_{C_w} \mathrm{d}w\, e_n(z)e_m(w)R(T(z)T(w))_q \end{aligned} \tag{14}$$

where the contours $C_w$ envelop the point $w$, and $R$ denotes the radial ordering

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |z| < |w|. \end{cases} \tag{15}$$

As is well known, although the OPEs on $\Sigma$ are generally $g$ dependent, the singularities of the OPEs on $\Sigma$ are $g$ independent. So that when $z$ is close to $w$, the OPE on $\Sigma$ for the generators of the $q$-KN algebra on $\Sigma$ is the same as the $g=0$ case

$$(T(z)T(w))_q = \frac{1}{z-w}\left(\frac{T(wq^{-1})}{zq-wq^{-1}} + \frac{T(wq)}{zq^{-1}-wq}\right) + \frac{1}{z-w} D^q_w T(w) + \text{regular terms} \tag{16}$$

where $D^q_w$ is the $q$-derivative

$$D^q_w f(w) = \frac{f(wq) - f(wq^{-1})}{w(q-q^{-1})}. \tag{17}$$

Making use of this definition for the $q$-derivative, one can rewrite the OPE (16) as

$$(T(z)T(w))_q = \frac{1}{w(q-q^{-1})}\left\{\frac{T(qw)}{z-wq^2} - \frac{T(q^{-1}w)}{z-wq^{-2}}\right\} + \text{regular terms} \tag{18}$$

which indicates that the OPE is singular at the points $z=wq^{\mp 2}$. Substituting (18) into (14), we have

$$
\begin{aligned}
[L_n, L_m] &= \oint_{C_\tau} \mathrm{d}w \oint_{C_w} \mathrm{d}z \frac{e_n(z)e_m(w)}{w(q-q^{-1})} \left\{ \frac{T(qw)}{z-wq^2} - \frac{T(q^{-1}w)}{z-wq^{-2}} \right\} \\
&= \frac{1}{q-q^{-1}} \oint_{C_\tau} \mathrm{d}w \Big\{ q^{n-m} e_n(wq)e'_m(wq)T(qw) \\
&\quad - q^{-(n-m)} e'_n(q^{-1}w)e_m(q^{-1}w)T(q^{-1}w) \Big\} \\
&= \frac{1}{q-q^{-1}} \sum_{r=-g_0}^{g_0} \{ q^{n-m} D^r_{nm} - q^{-(n-m)} D^r_{mn} \} L_{n+m-r} \\
&= \sum_{r=-g_0}^{g_0} \langle C^r_{nm} \rangle_q L_{n+m-r}
\end{aligned}
\tag{19}
$$

where the structure constant is given by

$$
\langle C^r_{nm} \rangle_q = \frac{q^{(n-m)} D^r_{nm} - q^{-(n-m)} D^r_{mn}}{q-q^{-1}} \tag{20}
$$

with

$$
D^r_{nm} = \oint_{C_\tau} \mathrm{d}w\, e_n(w)e'_m(w)\Omega_{n+m-r}(w). \tag{21}
$$

It is obvious that the $q$-deformed KN algebra coincides with the usual one [1] in the limit $q \to 1$.

## 3. The Ward $q$-identities for correlators

Consider a primary field $\Phi(z)$ with the conformal weight $h$. In quantum theory, the variation in $\Phi(z)$ is given by

$$
\delta\Phi(z) = \oint_{C_\tau} \mathrm{d}w\, \varepsilon(w) R(T(z)\Phi(w))_q \tag{22}
$$

where

$$
\varepsilon(z) = \sum_n \varepsilon_n e_n(z). \tag{23}
$$

The $q$-OPE in equation (22) is given by

$$
\begin{aligned}
&(T(z)\Phi(w))_q \\
&\quad = \frac{[\frac{1}{2}h]}{z-w} \left\{ \frac{\Phi(q^{-1}w)}{zq^{1/2h} - wq^{-1/2h}} + \frac{\Phi(qw)}{zq^{-1/2h} - wq^{1/2h}} \right\} + \frac{D^q_w \Phi(w)}{z-w} + \text{regular terms.}
\end{aligned}
\tag{24}
$$

In order to obtain the Ward $q$-identities for the correlation functions of primary fields on $\Sigma$, we consider the action of the generator of infinitesimal conformal transformations on the correlation of $n$ primary fields $\Phi_i(w_i)$ with correspondent conformal weights $h_i$ $(i=1,2,\ldots,n)$:

$$\left\langle \oint_{C_\tau} \mathrm{d}z\ \varepsilon(z)T(z)\Phi_1(w_1)\Phi_2(w_2)\ldots\Phi_n(w_n)\right\rangle_q \tag{25}$$

where $C_\tau$ encircles all the points $\{w_i q^{\pm h_i}, i=1,2,\ldots,n\}$. Here the correlation function $\langle\ldots\rangle_q$ is taken relative to the 'in' ($|0\rangle_q$) and the 'out' ($_q\langle 0|$) vacuums which are defined by requiring that

$$\begin{aligned} L_m|0\rangle_q&=0 \qquad m+g_0\geqslant -1\\ {}_q\langle 0|L_m&=0 \qquad m+g_0\leqslant 1. \end{aligned} \tag{26}$$

Note that the above conditions ensure the regularity of $T(z)|0\rangle_q$ and its adjoint at $z=0$ and $z=\infty$. By analyticity, the contour $C_\tau$ in (25) can be deformed to a sum of $n$ contours with each contour $C_{\tau_i}$ surrounding the points $\{w_i q^{\pm h_i}\}$. Then as a consequence of the $q$-OPE (24), we have

$$\begin{aligned} &\oint_{C_\tau} \mathrm{d}z\ \varepsilon(z)\langle T(z)\Phi_1(w_1)\ldots\Phi_n(w_n)\rangle_q\\ &\qquad=\sum_{i=1}^{n}\left\langle \Phi_1(w_1)\ldots\oint_{C\tau_i}\mathrm{d}z\ \varepsilon(z)(T(z)\Phi_i(w_i))_q\ldots\Phi_n(w_n)\right\rangle_q\\ &\qquad=\sum_{i=1}^{n}\oint_{C\tau_i}\mathrm{d}z\ \varepsilon(z)\hat{\mathscr{L}}^{h_i}_{z;w_i}\langle\Phi_1(w_1)\ldots\Phi_n(w_n)\rangle_q \end{aligned} \tag{27}$$

where the differential operator $\hat{\mathscr{L}}^{h_i}_{z;w_i}$ is given by

$$\hat{\mathscr{L}}^{h_i}_{z;w_i}=\frac{1}{z-w_i}\left\{[\tfrac{1}{2}h_i]\left(\frac{q^{-w_i\partial_{w_i}}}{zq^{1/2h_i}-w_iq^{-1/2h_i}}+\frac{q^{w_i\partial_{w_i}}}{zq^{-1/2h_i}-w_iq^{1/2h_i}}\right)+D^q_{w_i}\right\}. \tag{28}$$

We therefore obtain the Ward $q$-identity

$$\langle T(z)\Phi_1(w_1)\ldots\Phi_n(w_n)\rangle_q=\sum_{i=1}^{n}\hat{\mathscr{L}}^{h_i}_{z;w_i}\langle\Phi_1(w)\ldots\Phi_n(w_n)\rangle_q. \tag{29}$$

From equation (26), it is easy to see that the generators $L_{-g_0}$, and $L_{-g_0\pm 1}$ annihilate both the 'in' and 'out' vacuums. On substituting $\varepsilon(z)=e_m(z)$ for $m=-g_0$, $-g_0\pm 1$ into (27) and integrating, for any $n$-point function we obtain the following projective Ward $q$-identities:

$$\sum_{i=1}^{n} w_i^{-1}\{e_m(w_iq^{h_i})q^{w_i\partial_{w_i}}-e_m(w_iq^{-h_i})q^{-w_i\partial_{w_i}}\}\langle\Phi_1(w_1)\ldots\Phi_n(w_n)\rangle_q=0 \tag{30}$$

where $m=-g_0$, $-g_0\pm 1$.

It is well known that in standard conformal field theories the two-point and three-point functions are uniquely determined up to a normalization constant by the Ward identities. Nevertheless, the situation for the $q$-deformed case is quite different. Here the Ward $q$-identity do not uniquely specify them, as will be illustrated below. For this purpose we assume the correlation function of two primary fields $\Phi_1(w_1)$, $\Phi_2(w_2)$ with conformal weights $h_1$, $h_2$, respectively, to be of the form

$$\langle\Phi_1(w_1)\Phi_2(w_2)\rangle_q = \frac{1}{(w_1-w_2)_q^n} \qquad |w_1|>|w_2| \tag{31}$$

where the $q$-distance function is defined by

$$(w_1-w_2)_q^n = \sum_{k=1}^{n} \frac{[n]!}{[n-k]![k]!} w_1^{n-k}(-w_2)^k. \tag{32}$$

Substituting (31) into (30), for the $g\neq 0$ case we have the following conditions:

$$[h_1]=0=[h_2] \tag{33a}$$

$$[h_1-n-2g_0(h_1+h_2)]-[h_2-n-2g_0(h_1+h_2)]=0 \tag{33b}$$

$$[2h_1+h_2-n-2g_0(h_1+h_2)]-[2h_2+h_1-n-2g_0(h_1+h_2)]=0 \tag{33c}$$

$$[2g_0h_1+(2g_0-2)h_2+n]-[2g_0h_2+(2g_0-2)h_1+n]=0. \tag{33d}$$

When $|q|\neq 1$, these equations have only the trivial solution $h_1=0=h_2$ for all values of $n$. Obviously, this is an uninteresting case for physicists. However, when the deformation parameter $q$ is the root of unity, i.e. $q=e^{i\pi\alpha}$, they have the solution $h_1=h_2$ for all values of $n$. This means that $n$, which characterizes the solution, is not unique but arbitrary, so that the Ward $q$-identities do not determine the two-point function.

The above analyses hint that $q$-conformal field theory on higher genus Riemann surfaces may have some new features, except that the parameter $q$ is introduced through the deformation.

It will be of interest to study further features of $q$-conformal field theory on higher-genus Riemann surfaces.

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